Games, Trees, and Borel Functions

Brian Semmes

- Review of Baire space, Borel hierarchy, tree game
- Tree game characterization of Borel functions
- “Restricted” tree games
- Decomposing low-level Borel functions

Baire space

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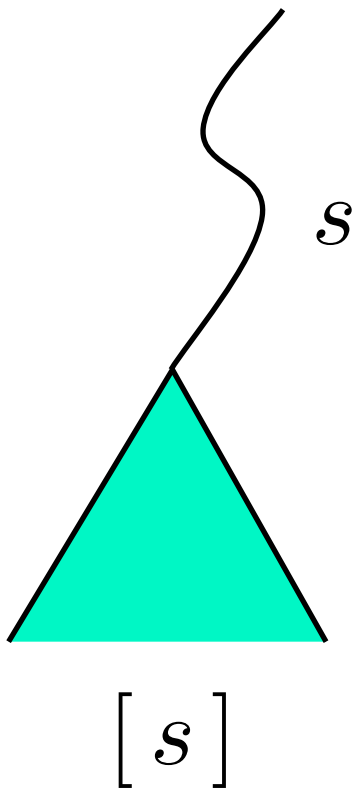

$$\mathbf{\Sigma}^0_1 := \{\bigcup_{n\in\omega} A_n : A_n \text{ basic open}\}$$

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$$\mathbf{\Sigma}^0_2 := \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_1\}$$

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$$\mathbf{\Sigma}^0_2 := \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_1\}$$

$$\vdots$$

$$\begin{array}{ccccccc}
 & & \mathbf{\Sigma}^0_1 & & & & \mathbf{\Sigma}^0_2 \\
 & \subset & & \supset & & \subset & \\
\mathbf{\Delta}^0_1 & & & & \mathbf{\Delta}^0_2 & & \cdots \\
 & \supset & & \subset & & \supset & \\
 & & \mathbf{\Pi}^0_1 & & & & \mathbf{\Pi}^0_2
\end{array}$$

The **Borel sets** are the smallest σ-algebra (closed under complements and countable unions) containing the open sets, i.e.

$$\text{Borel sets} = \bigcup_{\eta<\omega_1} \mathbf{\Sigma}^0_\eta$$

Let $f : {}^{\omega}\omega \longrightarrow {}^{\omega}\omega$.

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Say that f is **continuous** if $f^{-1}[Y] \in \mathbf{\Sigma}^0_1$ for every $Y \in \mathbf{\Sigma}^0_1$,

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f is **Baire Class $\boldsymbol{\alpha}$** if f is the pointwise limit of a sequence of functions f_n of Baire class $\xi_n < \alpha$ (where $\alpha < \omega_1$ and Baire class $0 =$ continuous),

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f is **Baire Class $\boldsymbol{\alpha}$** if f is the pointwise limit of a sequence of functions f_n of Baire class $\xi_n < \alpha$ (where $\alpha < \omega_1$ and Baire class $0 =$ continuous),

f is **Borel** if $f^{-1}[Y]$ is Borel for every Borel (equiv. $\mathbf{\Sigma}^0_1$) set Y.

Theorem. A function $f : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is Baire class α if and only if $f^{-1}[Y] \in \mathbf{\Sigma}^0_{\alpha+1}$ for every $Y \in \mathbf{\Sigma}^0_1$.

Theorem. A function $f : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is Baire class α if and only if $f^{-1}[Y] \in \mathbf{\Sigma}^0_{\alpha+1}$ for every $Y \in \mathbf{\Sigma}^0_1$.

Theorem (Lebesgue, Hausdorff, Banach). The Borel functions are the smallest class of functions containing the continuous functions and closed under pointwise limits, i.e.

$$\text{Borel functions} = \bigcup_{\alpha<\omega_1} \text{Baire class } \alpha$$

The Wadge game $G_{\mathrm{W}}(f)$

I

II

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0
II	

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0	
II		y_0

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0	x_1
II	y_0	

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0		x_1	
II		y_0		y_1

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0	x_1	x_2
II	y_0	y_1	

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0		x_1	x_2
II		y_0	y_1	

PASS!

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0	x_1	x_2
II	y_0	y_1	

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0	x_1	x_2	x_3
II	y_0	y_1		

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0	x_1	x_2	x_3
II	y_0	y_1		

PASS!

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0	x_1	x_2	x_3
II	y_0	y_1		

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0		x_1		x_2		x_3		x_4
II		y_0		y_1					

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0		x_1		x_2		x_3		x_4
II		y_0		y_1		y_2			

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0		x_1		x_2	x_3	x_4		$\dots$
II		y_0		y_1		y_2			

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0		x_1		x_2		x_3	x_4	$\cdots$	x
II		y_0		y_1		y_2				y

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0		x_1		x_2		x_3		x_4				x
II		y_0		y_1		y_2				$\cdots$			y

II wins iff y is infinite and $f(x) = y$

The Wadge game $G_{\mathrm{W}}(f)$

I	x_0		x_1		x_2		x_3		x_4				x
II		y_0		y_1		y_2				$\cdots$			y

II wins iff y is infinite and $f(x) = y$

II has a winning strategy iff
f is continuous

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I

II

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I x_0

II

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0	
II		y_0

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0	x_1
II	y_0	

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0		x_1	
II		y_0		y_1

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0	x_1	x_2
II	y_0	y_1	

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0	x_1	x_2
II	y_0		

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0	x_1	x_2	x_3
II		y_0		

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0	x_1	x_2	x_3
II		y_0	y_1'	

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0	x_1	x_2	x_3	x_4
II		y_0	y_1'		

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0		x_1		x_2		x_3		x_4
II		y_0		y_1'		y_2			

The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0		x_1		x_2	x_3	x_4	$\dots$
II		y_0		y_1'		y_2		

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I	x_0		x_1		x_2		x_3		x_4				x
II		y_0		y_1'		y_2				$\cdots$			y

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I	x_0		x_1		x_2		x_3		x_4				x
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The eraser game $G_{\mathrm{e}}(f)$ (Duparc)

I	x_0		x_1		x_2		x_3		x_4				x
II		y_0		y_1'		y_2				$\cdots$			y

II wins iff y is infinite and $f(x) = y$

II has a winning strategy iff
f is Baire class 1

A set $T \subseteq {}^{<\omega}\omega$ is a **tree** if $s \in T$ and $u \subseteq s \Rightarrow u \in T$.

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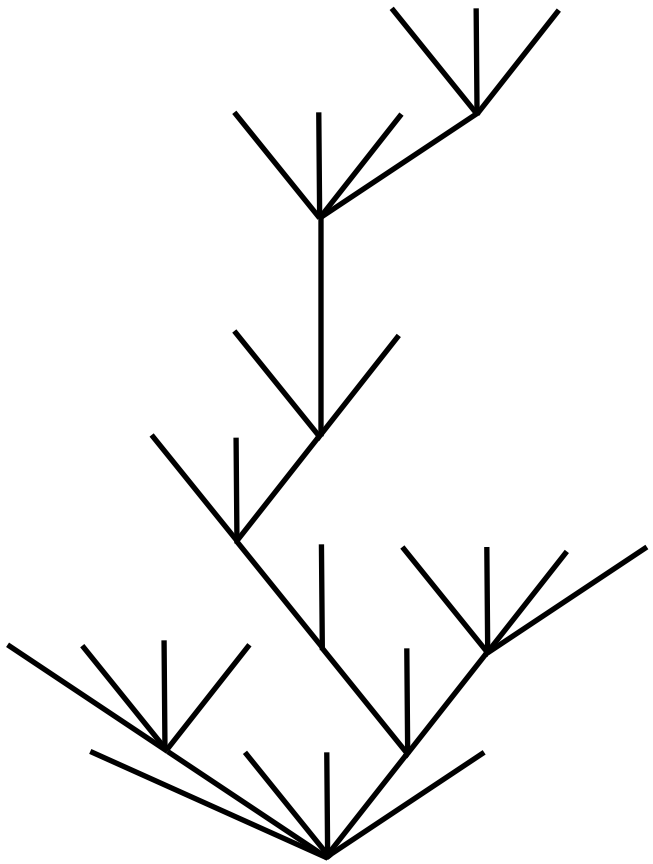

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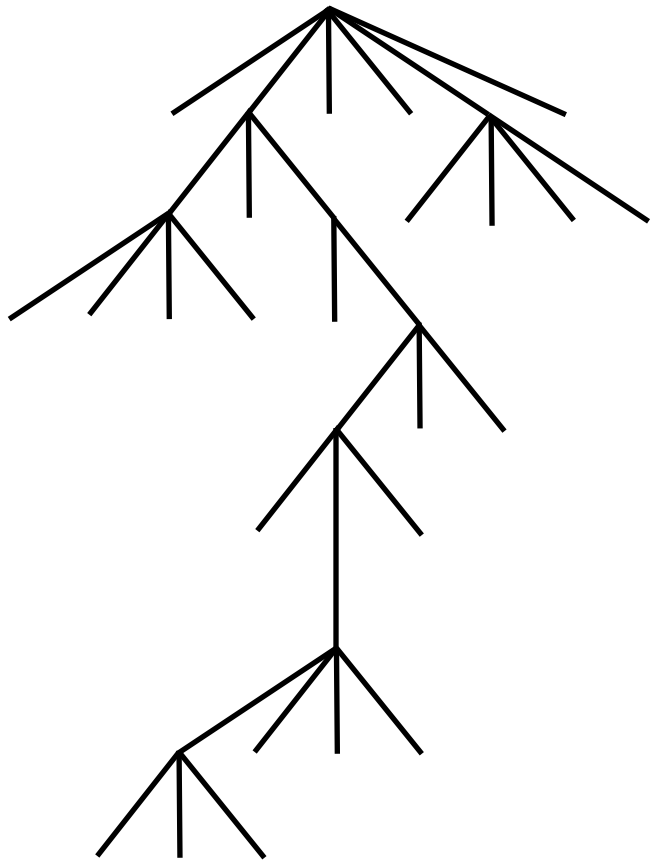

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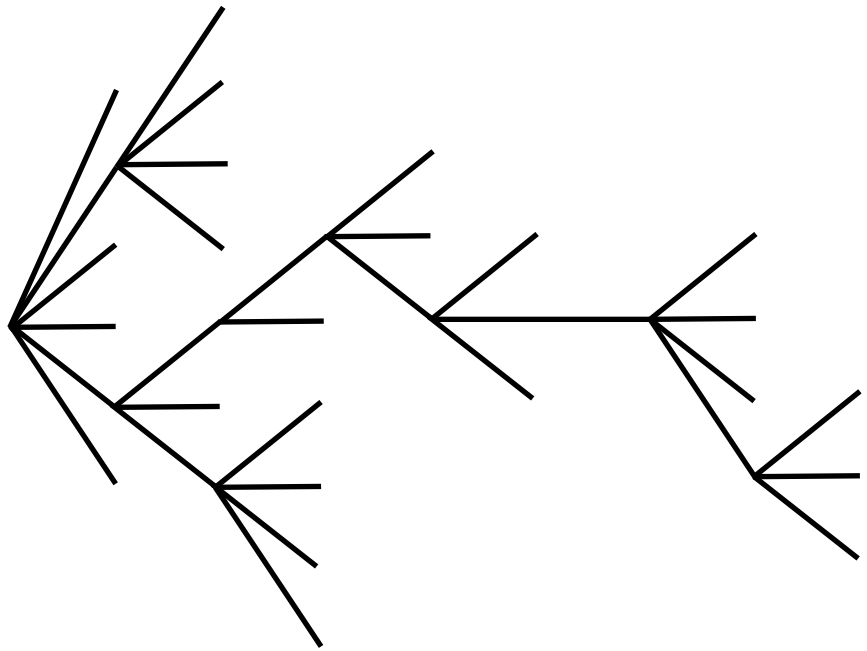

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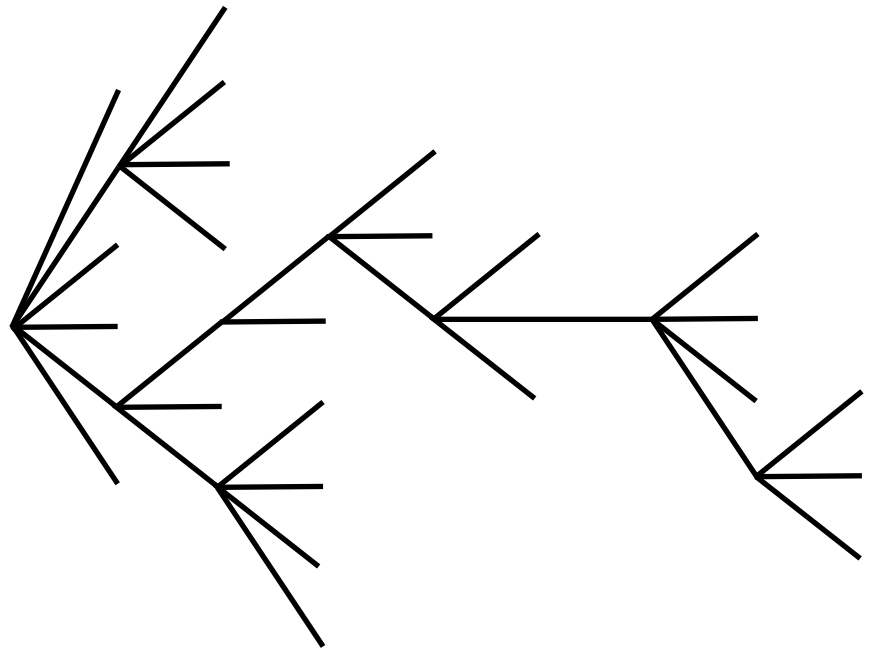


An element $x \in {}^{\omega}\omega$ is an $\boldsymbol{\infty}$ **branch** of T if for all $n \in \omega$, $x \restriction n \in T$.

Let $T \subseteq {}^{<\omega}\omega$ be a tree.

A **labelled tree** is a function $\phi : T \to {}^{<\omega}\omega$ that is monotone and length-preserving, i.e.

$s \subseteq u \Rightarrow \phi(s) \subseteq \phi(u)$ and

$\mathrm{lh}(\phi(s)) = \mathrm{lh}(s)$.

The tree game $G_{\mathrm{t}}(f)$

I

II

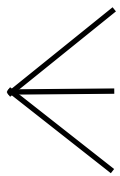
The tree game $G_{\mathrm{t}}(f)$

I	x_0
II	

The tree game $G_{\mathrm{t}}(f)$

I x_0

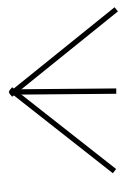
II



The tree game $G_{\mathrm{t}}(f)$

I x_0 x_1

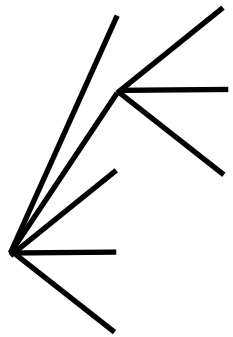
II



The tree game $G_{\mathrm{t}}(f)$

I x_0 x_1

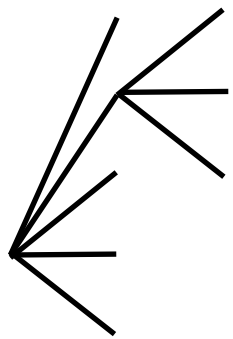
II



The tree game $G_{\mathrm{t}}(f)$

I x_0 x_1 x_2

II



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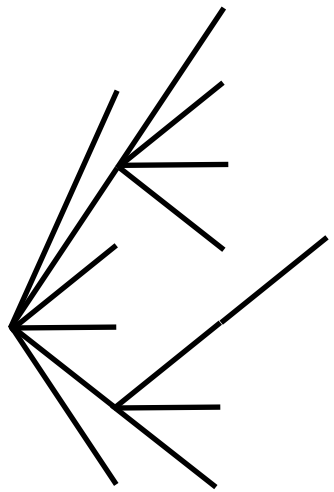
I x_0 x_1 x_2

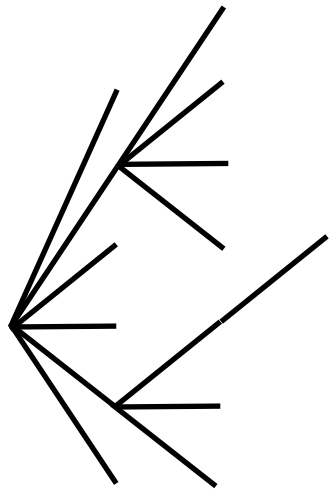
II



The tree game $G_{\mathrm{t}}(f)$

I x_0 x_1 x_2 x_3

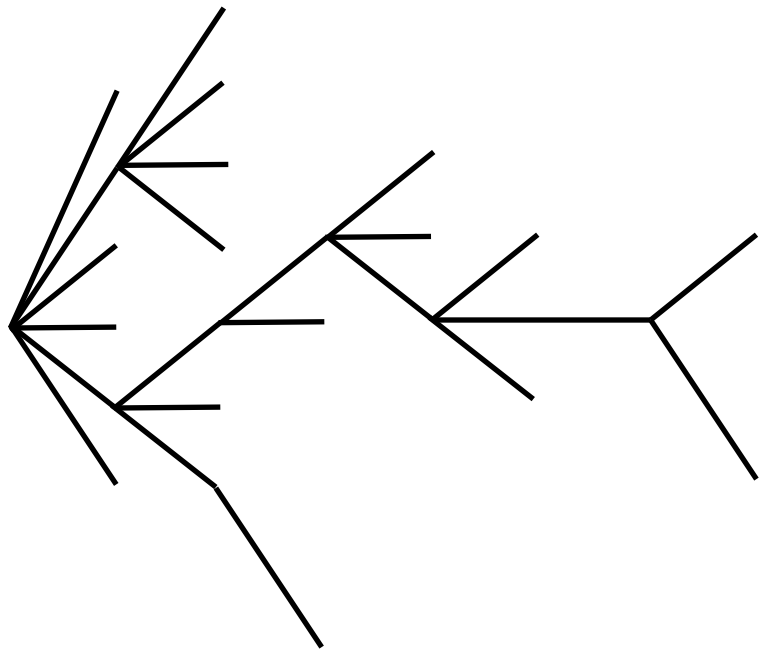
II



The tree game $G_{\mathrm{t}}(f)$

I x_0 x_1 x_2 x_3

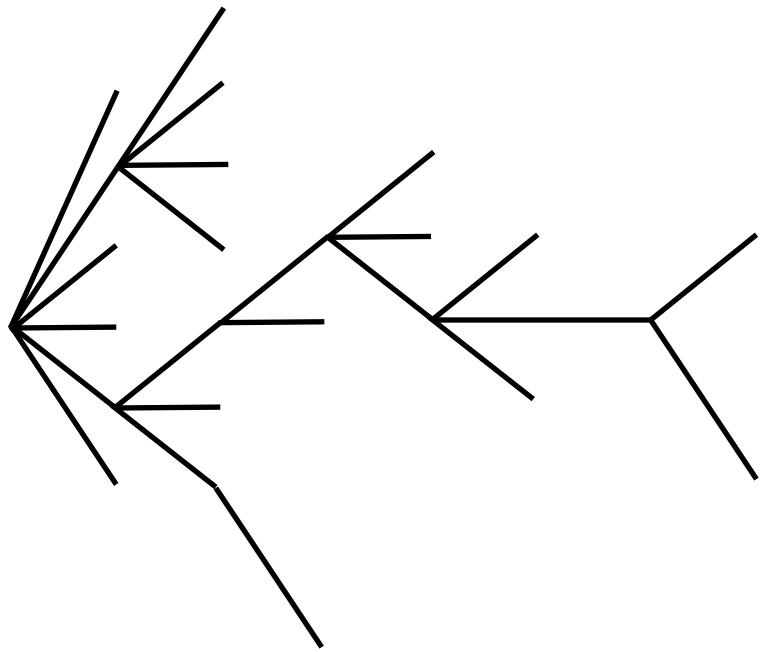
II



The tree game $G_{\mathrm{t}}(f)$

I x_0 x_1 x_2 x_3 x_4

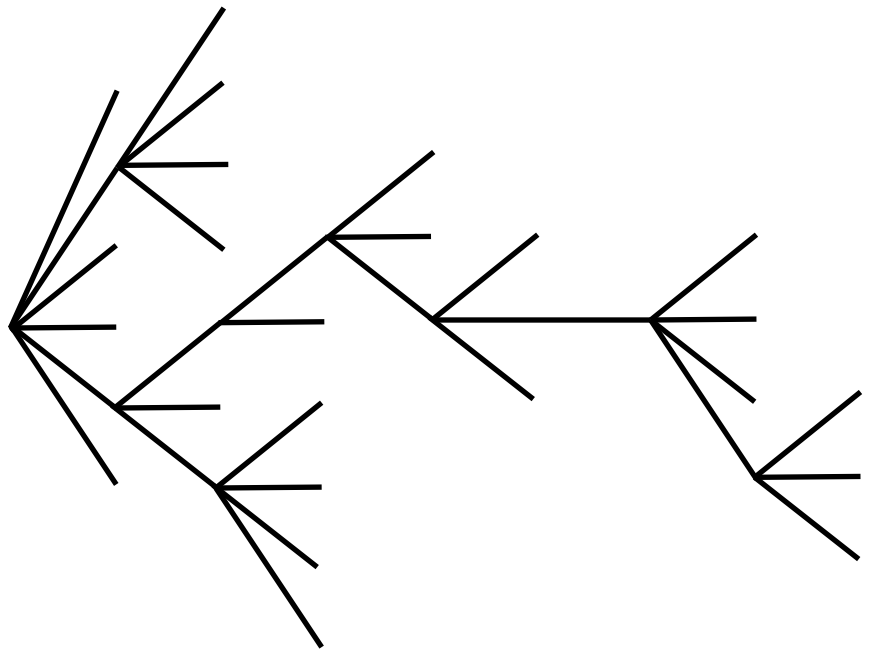
II



The tree game $G_{\mathrm{t}}(f)$

I x_0 x_1 x_2 x_3 x_4

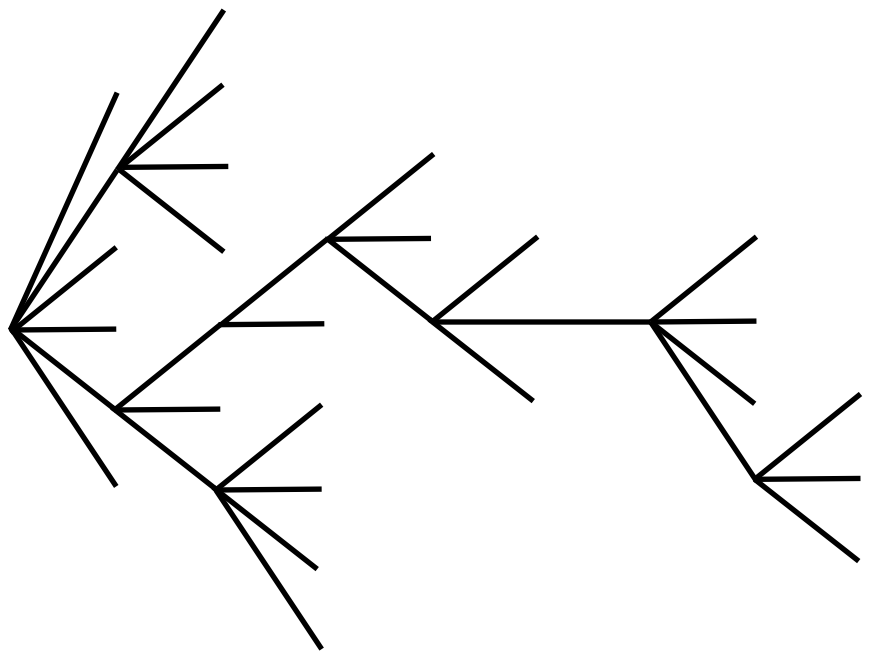
II



The tree game $G_{\mathrm{t}}(f)$

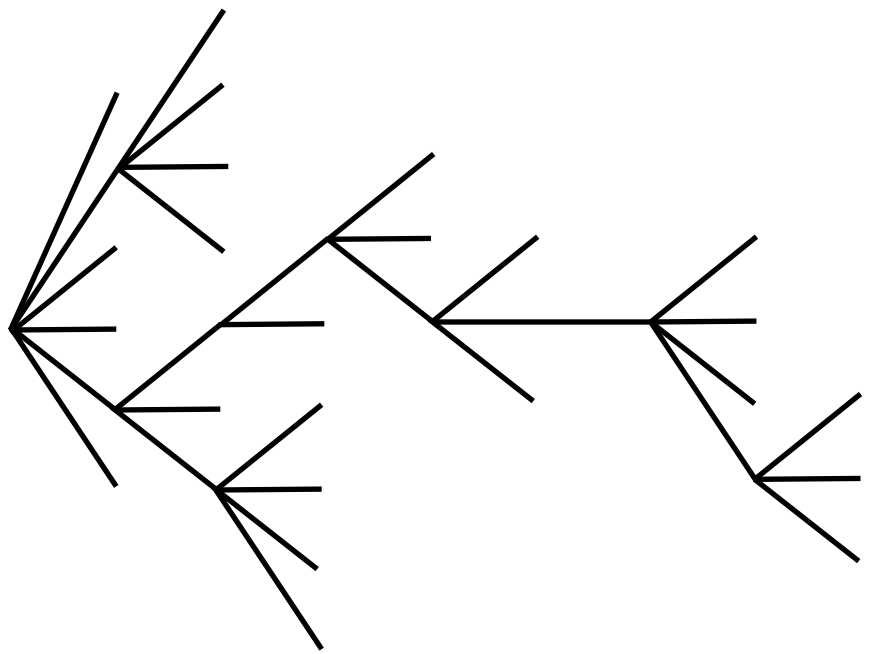
I x_0 x_1 x_2 x_3 x_4 $\ldots$

II



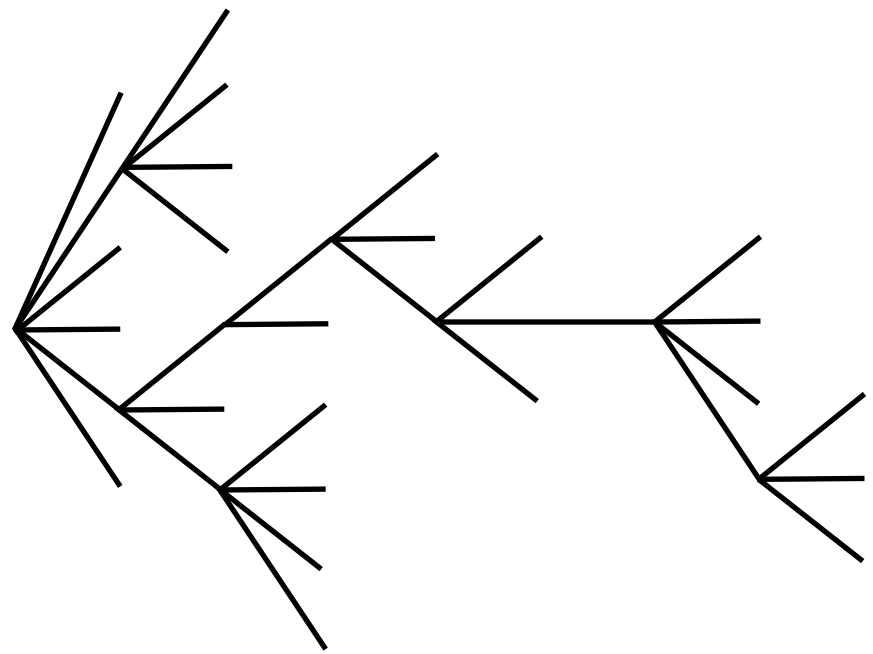
The tree game $G_{\mathrm{t}}(f)$

I x_0 x_1 x_2 x_3 x_4 $\cdots$ x

II  ϕ

The tree game $G_{\mathrm{t}}(f)$

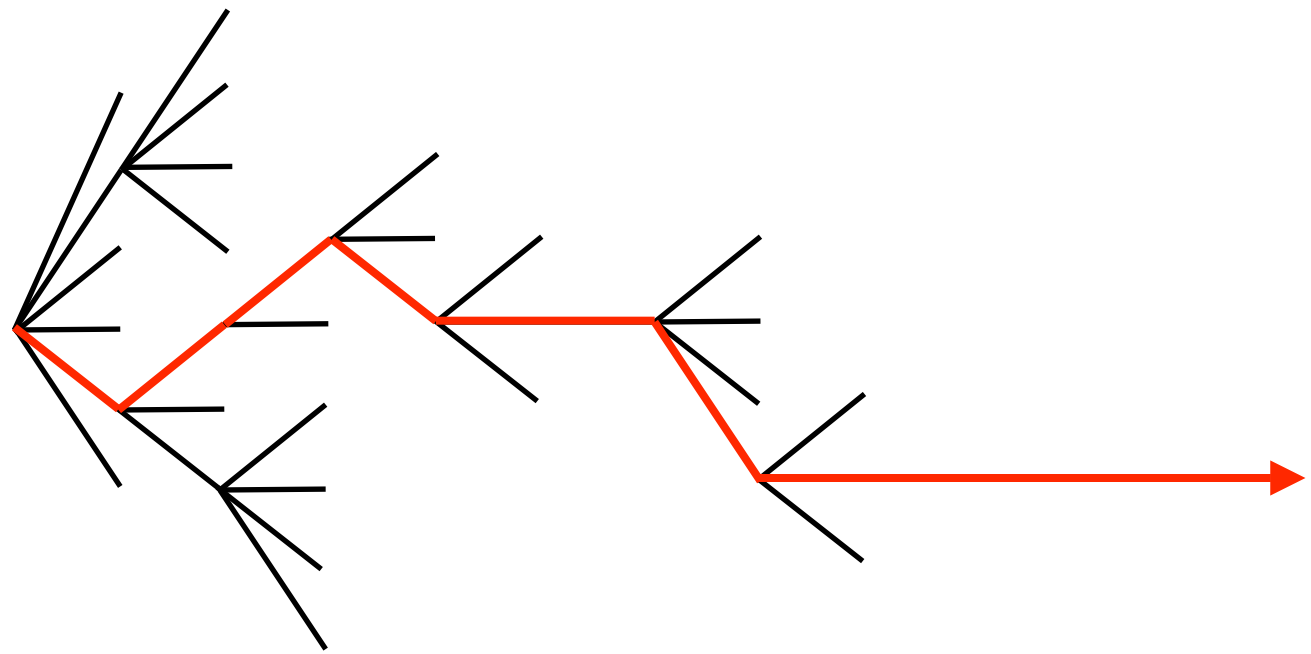
I	x_0		x_1		x_2		x_3		x_4		$\cdots$		x
II													ϕ



II wins if $\mathrm{dom}(\phi)$ has a unique ∞ branch z and $\bigcup_{n\in\omega} \phi(z \restriction n) = f(x)$

The tree game $G_{\mathrm{t}}(f)$

I	x_0		x_1		x_2		x_3		x_4		$\cdots$		x
II													ϕ



II wins if $\mathrm{dom}(\phi)$ has a unique ∞ branch z and $\bigcup_{n\in\omega} \phi(z \restriction n) = f(x)$

A function f is Borel iff
II has a winning strategy in $G_{\mathrm{t}}(f)$.

Proof idea: if f is the pointwise limit of f_n and τ_n is a winning tree strategy for II in $G_{\mathrm{t}}(f_n)$, construct a winning tree strategy τ for II in $G_{\mathrm{t}}(f)$.

Plan: restrict the tree game to characterize subclasses of Borel functions.

A function f is $\boldsymbol{n \longrightarrow m}$ if $f^{-1}[Y] \in \mathbf{\Sigma}^0_m$ for every $Y \in \mathbf{\Sigma}^0_n$.

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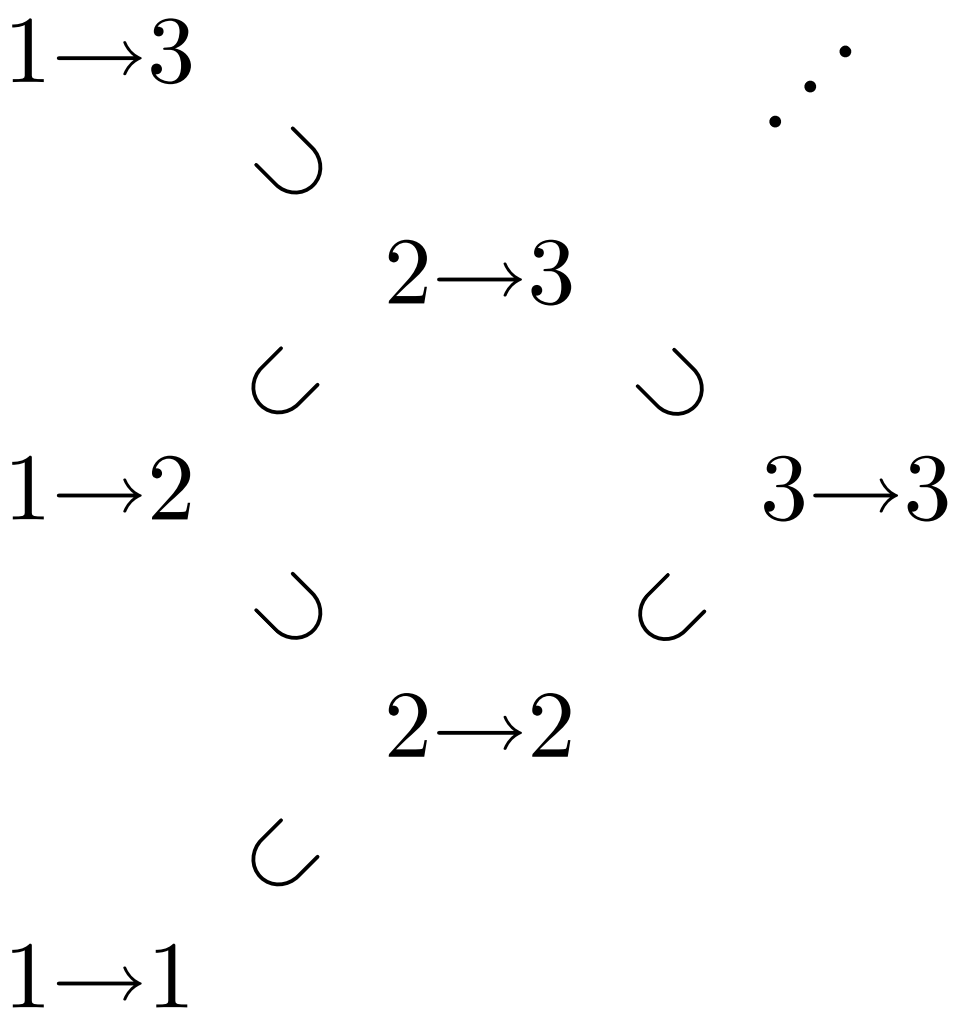

Tree games

Tree games

$1 \to 1$ (continuous):
II produces ϕ such that $\mathrm{dom}(\phi)$ is linear:
$s, t \in \mathrm{dom}(\phi) \Rightarrow s \subseteq t \vee t \subseteq s$

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$1 \to 1$ (continuous):
II produces ϕ such that $\mathrm{dom}(\phi)$ is linear:
$s, t \in \mathrm{dom}(\phi) \Rightarrow s \subseteq t \vee t \subseteq s$

$1 \to 2$ (Baire class 1):
II produces ϕ such that $\mathrm{dom}(\phi)$ is finitely branching: $s \in \mathrm{dom}(\phi) \Rightarrow \{s^\frown k \in \mathrm{dom}(\phi)\}$ is finite

$1 \to 3$ (Baire class 2):
II produces ϕ such that $\{s^\frown k \in \mathrm{dom}(\phi)\}$ is infinite $\Rightarrow s \subset z_\phi$

1→3 (Baire class 2):
II produces ϕ such that $\{s^\frown k \in \mathrm{dom}(\phi)\}$ is infinite $\Rightarrow s \subset z_\phi$

2→2:
II produces ϕ such that $\mathrm{dom}(\phi)$ is finitely branching at the root and linear thereafter

2→3:
II produces ϕ such that dom(ϕ) may branch infinitely at the root but is finitely branching thereafter

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II produces ϕ such that dom(ϕ) may branch infinitely at the root but is finitely branching thereafter

3→3:
II produces ϕ such that dom(ϕ) may branch infinitely at the root but is linear thereafter

Characterizations of $2 \to 2$, $2 \to 3$, and $3 \to 3$ yield decomposition theorems:

f is $2 \to 2 \Leftrightarrow$ there is a $\mathbf{\Pi}^0_1$ partition $\langle A_n : n \in \omega \rangle$ such that $f \restriction A_n$ is continuous (Jayne, Rogers 1982)

f is $2 \to 3 \Leftrightarrow$ there is a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega \rangle$ such that $f \restriction A_n$ is $1 \to 2$

f is $3 \to 3 \Leftrightarrow$ there is a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega \rangle$ such that $f \restriction A_n$ is continuous

f is $3 \to 3 \Leftrightarrow$ there is a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega \rangle$ such that $f \restriction A_n$ is continuous

Proof idea for $2 \to 2$ ($\Rightarrow$):
suppose there is no winning backtrack strategy for II, use II's winning eraser strategy to recursively construct $\mathbf{\Sigma}^0_1$ set Y such that $f^{-1}[Y]$ is $\mathbf{\Sigma}^0_2$-complete.

Proof idea for $2 \rightarrow 3$ ($\Rightarrow$):
suppose there is no winning $2 \rightarrow 3$ strategy for II, use II's winning $1 \rightarrow 3$ strategy to recursively construct $\mathbf{\Sigma}^0_1$ set Y such that $f^{-1}[Y]$ is $\mathbf{\Sigma}^0_3$-complete.

Proof idea for 3→3 ($\Rightarrow$):
suppose there is no winning multitape strategy for II, use II's winning 2→3 strategy to recursively construct $\mathbf{\Sigma}^0_2$ set Y such that $f^{-1}[Y]$ is $\mathbf{\Sigma}^0_3$-complete.